Mirror Symmetry and Enumerative Geometry

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What is Enumerative Geometry?

Enumerative geometry is about counting geometric objects. It has been studied for many centuries. For example:

1. How many times do two curves touch? (Bézout's theorem, first stated ~1690 by Newton)
2. How many circles are tangent to three given circles? (8, Apollonius's problem, first studied ~200BC)
3. How many straight lines can be drawn on a surface defined by a degree 3 polynomial? (27, shown by Cayley and Salmon, 1849)
4. How many curves* defined by degree d polynomials pass through $3d-1$ points? (Cases $d=1$, $d=2$ known since antiquity, $d=3$ and $d=4$ found in the 19th century)
   (*rational curves)

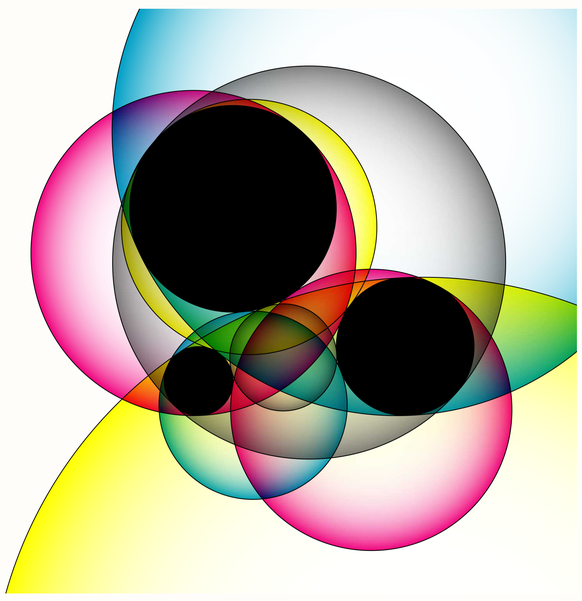

Figure 1: A solution to Apollonius's problem – the eight colourful circles are each tangent to all three black circles.[4]

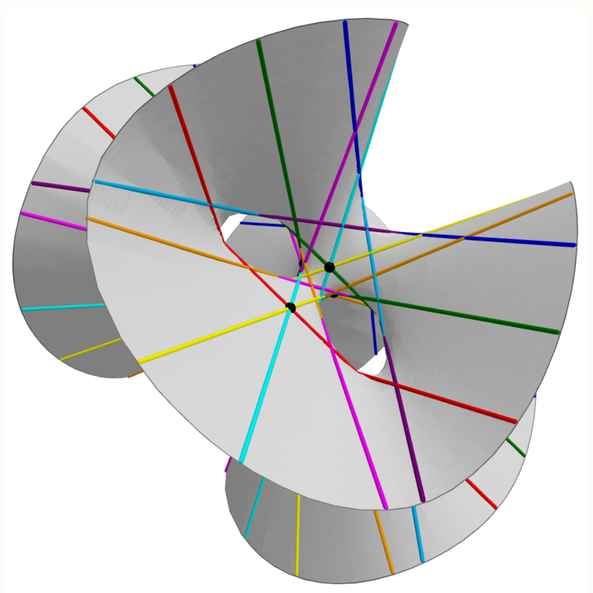

Figure 2: A cubic surface with its 27 lines.[3]

Curve Counting in Algebraic Varieties

Varieties are geometric spaces. They can be curved, and don't have to be smooth (think of a pineapple!).
Gromov-Witten invariants are numbers counting how many curves pass through certain parts of a variety. They appear a lot in modern enumerative geometry, but are very hard to calculate.

Kontsevich's Formula

In 1995, Kontsevich [2] found a recursive solution to problem (4) above using Gromov-Witten invariants. The formula is:

$$N_d = \sum_{d_1+d_2=d} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

What is Mirror Symmetry?

Mirror symmetry is a relationship between the curve counts (Gromov-Witten invariants) of a given variety X, and seemingly unrelated(!) aspects of the geometry of a different variety $\check{X}$.
The relevant geometric properties of the "mirror variety" $\check{X}$ are determined by integrals of differential forms, which makes it (relatively) easy to calculate Gromov-Witten invariants.

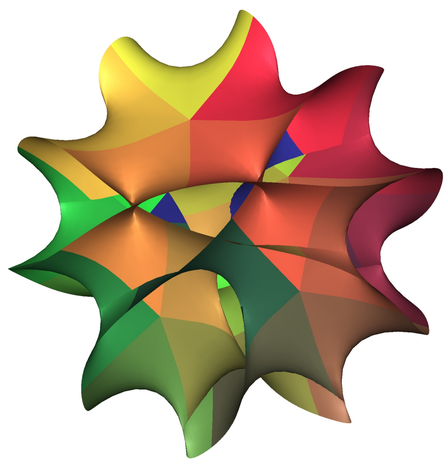

Figure 3: Mirror symmetry was first developped for the ***quintic threefold***, a 3D variety of which a 2D slice is shown here.[1]

Toric Varieties

So far, most results about mirror symmetry have only been proven for ***toric varieties*** – varieties with lots of symmetry, whose geometry can be studied with linear algebra and combinatorics.
If a variety is not toric, we must deform it into a toric variety before we can use mirror symmetry (a hard and technical process which is not always possible!).

Current Work & Future Directions

I am currently working on generalising some results from toric varieties to ***toric variety bundles*** – a kind of variety which combines a toric variety with a non-toric one (like how a cylinder or a Möbius strip combines a circle with a line segment).
In the future I hope to also look into ***Hodge atoms*** – a way of turning Gromov-Witten invariants into structures which can classify* algebraic varieties.
(*up to birational equivalence)

References

[1] Andrew J. Hanson. Wikimedia Commons. CC BY-SA 3.0.

[2] Maxim Kontsevich. "Enumeration of Rational Curves Via Torus Actions". In: ***The Moduli Space of Curves***. Boston, MA: Birkhäuser Boston, 1995, pp. 335–368. ISBN: 978-1-4612-8714-8.

[3] Paul Masson. Wikimedia Commons. MIT License.

[4] Melchoir. Wikimedia Commons. CC BY-SA 3.0.