Toric Varieties

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Abstract

We discuss the basics of the theory of smooth toric varieties and how the combinatorial structure of their fans encodes information about their geometry. To study the fan structure, we use the concepts of primitive collections and primitive relations introduced by Batyrev. The main results are the characterisation of toric projective bundles and the classification of toric Fano 4-folds with Picard number 2.

Contents

1 Introduction

Toric varieties are a special case of algebraic varieties that, despite being not at all general, provide a "remarkably fertile testing ground for new theories" [Ful]. The ideals that define affine toric varieties arise from simple monomial equations, and for abstract toric varieties the local coordinates of any affine component can be expressed as monomial functions of the local coordinates of another. This makes them a good source of examples of algebro-geometric phenomena, as noted by Mumford in the introduction to [KKMSD].

Toric varieties correspond to combinatorial objects called *fans* which encode a remarkable amount of geometric information. This allows the study of toric varieties via purely combinatorial means, as for example in Batyrev's work on classification which we discuss later in the report. These fans live inside real vector spaces, so are easy to visualise and even draw in lower dimensions, making their study all the easier. Many toric varieties also correspond to lattice polytopes, and the study of polytopes has impacts in the study of toric varieties and vice versa.

In Section 2 we cover the basics of the theory of toric varieties, their fans and their polytopes. We also discuss the relationship between cones and torus orbits, and how torus-invariant divisors give an alternative construction of the Picard group. We also touch on sheaves, in anticipation of Section 4.1.

Section 3 introduces *primitive collections* and their corresponding *primitive relations* – concepts introduced by Batyrev in [Bat1] to facilitate the study of a fan's combinatorial structure. In Section 4 we use these concepts to determine when one toric variety is a projective bundle over another.

In Section 5, we again use the language of Section 3, this time applying it to lattice polytopes. We see how primitive relations determine isomorphism classes of toric Fano varieties, and classify toric Fano 4-folds with Picard number 2 up to isomorphism, in line with the results of [Bat2].

The main sources of definitions and results are the textbooks on toric varieties by William Fulton [Ful] and by Cox, Little, and Schenck [CLS], whose results are used throughout.

Notation and Conventions

All varieties are considered to be irreducible. We use $\mathbb{P}^k$ to refer to k-dimensional complex projective space, and $\mathbb{N}$ to refer to the set of nonnegative integers $\{0, 1, 2, \ldots\}$. We use the word *lattice* in the sense of "free abelian group"; for a finite set A we write $\mathbb{Z}A$ for "the lattice with basis A". The symbol $\Diamond$ marks the end of an example.

2 Background

We begin by defining the main geometric object of study.

Definition 2.1. An *algebraic torus* is an algebraic group that is isomorphic to $(\mathbb{C}^*)^n$ for some natural number n. A *toric variety* is a normal algebraic variety X that contains an algebraic torus T as a dense open subset, along with a group action of T on X (given by a morphism $T \times X \to X$) which extends the action of T on itself.

One of the great advantages of studying toric varieties is the ability to translate geometric properties into combinatorial ones. The combinatorial objects we use for this purpose are called *fans*, which themselves are made up of *cones*. Some authors omit the assumption of normality when defining a toric variety, but in this report we only consider the case of normal, separated toric varieties, and these two adjectives are typically omitted throughout. This is motivated by the following correspondence, shown in [CLS, §3.1]:

$$\left\{ \begin{array}{c} \text{Normal separated irreducible} \\ \text{toric varieties with torus } (\mathbb{C}^*)^n \end{array} \right\} \longleftrightarrow \{\text{Fans in } \mathbb{R}^n\}.$$

In other words, each toric variety has a unique corresponding fan, which we can use to learn about the variety's geometric properties.

2.1 Cones and Fans

In this subsection we work in fixed dimension $n \in \mathbb{N}$.

Definition 2.2. Let $k \in \mathbb{N}$. A *strongly convex rational polyhedral cone* (or simply a *cone*) is a subset

$$\sigma = \operatorname{Cone}(u_1, \ldots, u_k) = \{\lambda_1 u_1 + \ldots + \lambda_k u_k \mid \lambda_i \geq 0\} \subseteq \mathbb{R}^n \tag{1}$$

satisfying the following conditions:

1. (*Rationality*) For any $1 \leq i \leq k$, the vector $u_i \in \mathbb{R}^n$ belongs to the lattice $\mathbb{Z}^n$.
2. (*Strong convexity*) $\sigma \cap -\sigma = \{0\}$, where $-\sigma = \{-u \mid u \in \sigma\}$.

We call $u_1, \ldots, u_k$ the *generators* of σ and use the convention $\operatorname{Cone}(\varnothing) = \{0\}$. A cone σ is *regular* (or *smooth* in the language of [CLS]) if its generators form a subset of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. When speaking of the generators of a cone we usually mean the smallest possible generating set, and assume that these generators have minimal norm.[1]

[1] A minimal generating set for a cone is unique up to scaling each generator by a positive real number, so the requirement of minimal norm (amongst lattice points) gives us a unique generating set for each cone.

It is convenient to restrict ourselves to regular cones – and thus to smooth toric varieties – for which the concepts of dimension and faces have very simple definitions.

Definition 2.3. Let σ be a regular cone and let $S = \{u_1, \ldots, u_k\} \subseteq \mathbb{Z}^n$ be the set of generators of σ. We call k the *dimension* of σ and write $\dim \sigma = k$. We will often consider *maximal* cones – we mean this in the sense of dimension.

A subset $\tau \subseteq \sigma$ is a *face* of σ if we have $\tau = \text{Cone}(A)$ for some $A \subseteq S$. In this case we write $\tau \preceq \sigma$. If $\dim \tau = \dim \sigma - 1$ then we call τ a *facet* of σ, and if $\dim \tau = 1$ then we call τ an *edge* of σ. More generally, a cone of dimension 1 is called a *ray*.

An important result is that generators of any cone σ are precisely the generators of the edges of σ.

Definition 2.4 ([Bat1], definition 2.4)**.** A finite collection $\Sigma = \{\sigma_1, \ldots \sigma_k\}$ of regular cones in $\mathbb{R}^n$ is called a *complete regular n-dimensional fan* (or simply a *fan*) if it satisfies the following conditions:

(i) If $\sigma \in \Sigma$ and $\tau \preceq \sigma$ then $\tau \in \Sigma$.

(ii) If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \preceq \sigma_1$ and $\sigma_1 \cap \sigma_2 \preceq \sigma_2$.

(iii) (Completeness) $\mathbb{R}^n = \bigcup_{i=1}^k \sigma_i$.

We denote by $G(\Sigma)$ the set of generators of all $\sigma \in \Sigma$.

Remark 2.5. The first two conditions in 2.4 are stated more clearly in [Ful]:

(i) Any face of a cone in the fan is itself a cone in the fan.

(ii) The intersection of any two cones in the fan is a face of each.

The completeness condition simply states that the cones of the fan cover the ambient space.

From here on we require all our non-affine toric varieties to be *complete* – i.e. to come from a complete fan – so all fans are assumed complete unless otherwise stated.

In general, we can define cones and fans on any rank n lattice $N \cong \mathbb{Z}^n$ and the corresponding real vector space $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$ exactly as above. We distinguish $N \cong \mathbb{Z}^n$ from its dual lattice $M = \text{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$, and given a cone $\sigma \subseteq N_\mathbb{R}$ we use the dual cone $\sigma^\vee \subseteq M_\mathbb{R}$ to define an affine toric variety. For this purpose it can be useful to think of M as being made up of monomials $\chi^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ (for some $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$).

2.2 The Toric Variety of a Fan

Classically, an affine variety V is defined as the zero locus of the polynomials in some prime ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$. A better definition for us uses the following fact: any point $p \in V$ corresponds to a maximal ideal of the quotient ring $\mathbb{C}[V] = \mathbb{C}[x_1, \ldots, x_n]/I$, given by $\{f \in \mathbb{C}[V] \mid f(p) = 0\}$, and more generally any subvariety of V corresponds to a prime ideal of this ring. This allows us to consider V as the set of prime ideals of $\mathbb{C}[V]$, denoted by $\operatorname{Spec}\mathbb{C}[V]$, of which the maximal ideals are the closed points. We call $\mathbb{C}[V]$ the *coordinate ring* of V and think of its elements as functions on V, called *regular functions*.

The coordinate ring of an affine variety is an integral domain and a finitely-generated $\mathbb{C}$-algebra. In fact, we can use Spec to associate an affine variety with any such $\mathbb{C}$-algebra.

Another way of creating these $\mathbb{C}$-algebras is via semigroups. For our purposes, a *semigroup* is a set with an identity and an associative binary operation. We call a semigroup *affine* if it is finitely generated, if the binary operation is commutative (we denote this operation by $+$) and if it can be embedded in a lattice. For an affine semigroup S with generating set A we write $\mathsf{S} = \mathbb{N}A$. Given an affine semigroup $\mathsf{S} = \mathbb{N}A$ with $A = \{u_1, \ldots, u_k\}$ we define the corresponding $\mathbb{C}$-algebra $\mathbb{C}[\mathsf{S}] = \mathbb{C}[\chi^{u_1}, \ldots, \chi^{u_k}]$ with multiplication given by $\chi^{u_i}\chi^{u_j} = \chi^{u_i+u_j}$.

We can create semigroups from cones. To do this we need the concept of the dual of a cone.

Definition 2.6. Let σ be a cone in $N_\mathbb{R}$ and let $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$ be the dual pairing. The *dual cone* of σ is

$$\sigma^\vee = \{u \in M_\mathbb{R} \mid \langle u, v \rangle \geq 0 \quad \forall v \in \sigma\}. \tag{2}$$

In particular, $\sigma^\vee$ is a (not necessarily strongly convex) cone in $M_\mathbb{R}$ and $(\sigma^\vee)^\vee = \sigma$. If we consider both $M_\mathbb{R}$ and $N_\mathbb{R}$ as $\mathbb{R}^n$ then the dual pairing is just the dot product and $\sigma^\vee$ is generated by the facet normals of σ.

Given a cone σ in $N_\mathbb{R}$, the set $\sigma^\vee \cap M$ is a semigroup under the addition induced by M. It clearly satisfies the second and third conditions for being an affine semigroup, and *Gordan's lemma* (see [Ful, p12]) tells us that it is also finitely generated. In fact, in the case of regular cones we can do even better.

Proposition 2.7. *Let σ be a regular cone in $N_\mathbb{R}$. Then the generators of $\sigma^\vee$ also generate the affine semigroup $\sigma^\vee \cap M$.*

Proof. Let $A = \{u_1, \ldots u_k\}$ be the generating set of $\sigma^\vee$ and let $u \in \sigma^\vee \cap M = \mathsf{S}$. Then there exist $\lambda_1, \ldots, \lambda_k \geq 0$ such that $u = \lambda_1 u_1 + \cdots + \lambda_k u_k$. The cone $\sigma^\vee$ is regular as σ is regular (see [CLS, p30]), so A is a subset of a $\mathbb{Z}$-basis of M, and so we can take $\lambda_1, \ldots, \lambda_k \in \mathbb{Z}$. Hence $\mathsf{S} = \{\lambda_1 u_1 + \cdots + \lambda_k u_k \mid \lambda_i \in \mathbb{N}\}$ i.e. $\mathsf{S} = \mathbb{N}A$. □

We can now turn cones into affine varieties[2] by taking $\operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap M])$, and we denote this variety by U_σ. This is indeed a toric variety (see [CLS, Proposition 1.1.14]) with torus $(\mathbb{C}^*)^{\dim U_\sigma}$.

[2] In fact, all affine toric varieties arise this way.

However, not all varieties are affine. *Abstract varieties* are constructed by gluing together affine varieties along shared open subsets, expanding on the fact that all quasi-projective varieties admit an open covering of affine varieties. It turns out that fans are precisely the object we need to determine how to glue our affine toric varieties.

Any two cones σ_1, σ_2 in a fan Σ have at least one common face, and there is a maximal such face $\tau = \sigma_1 \cap \sigma_2 \in \Sigma$ by the definition of a fan. This implies that U_{σ_1} and U_{σ_2} respectively contain open subsets U_1, U_2 that are each isomorphic to U_τ. The *gluing isomorphism* $g_{12} : U_1 \to U_2$ gives an equivalence relation $\sim$ defined as follows: for $x \in U_1$ and $y \in U_2$, we have $x \sim y$ if and only if $y = g_{12}(x)$. Then the abstract variety obtained from the gluing is the space

$$X = (U_{\sigma_1} \sqcup U_{\sigma_2}) / \sim$$

equipped with the quotient topology.

We can extend the equivalence relation as $x \sim y \Leftrightarrow y = g_{ij}(x)$ for $\sigma_i \neq \sigma_j \in \Sigma$, and glue all the cones of Σ together as above to get the toric variety

$$X_\Sigma = \left(\bigsqcup_{\sigma \in \Sigma} U_\sigma \right) \Big/ \sim .$$

Note that $\{0\} \subseteq N_\mathbb{R}$ is a face of every cone in Σ, and $U_{\{0\}} = (\mathbb{C}^*)^n$ (where n is the rank of N), so our gluing always preserves the algebraic torus. The torus actions of each affine component are compatible as g_{ij} induces the identity map on $\mathbb{C}[(\sigma_i \cap \sigma_j)^\vee \cap M)]$ and so these actions can be patched together to obtain a torus action on X_Σ. Indeed, X_Σ is a normal separated toric variety as shown in [CLS, Theorem 3.1.5].

2.3 Example: Projective Space as a Toric Variety

A simple example of a non-affine toric variety is complex projective space $\mathbb{P}^n$. The fan of $\mathbb{P}^n$ lives in $N_\mathbb{R} \cong \mathbb{R}^n$ and is generated by $\ell_0, \dots \ell_n$ where $\{\ell_1, \dots, \ell_n\}$ is a $\mathbb{Z}$-basis of N and $\ell_0 = -(\ell_1 + \cdots + \ell_n)$. The maximal cones of the fan are generated by the cardinality-n subsets of $\{\ell_0, \dots \ell_n\}$. In this section we construct $\mathbb{P}^2$ from its fan following the method described in Section 2.2.

Figure 1 shows the fan of $\mathbb{P}^2$ and the duals of its maximal cones. Note that despite the fan of $\mathbb{P}^2$ being complete, the dual cones do not cover $M_\mathbb{R}$. In fact, there is no reason to expect the dual cones to form a fan at all. This is why we focus on cones in $N_\mathbb{R}$ instead of working exclusively in the dual space.

The construction is made easier by the following result.

Lemma 2.8 (Separation Lemma [CLS, Proposition 3.1.3])**.** *Let σ_1, σ_2 be two cones in a fan Σ and let $\tau = \sigma_1 \cap \sigma_2$. Then*

$$\tau^\vee \cap M = \sigma_1^\vee \cap M + \sigma_2^\vee \cap M.$$

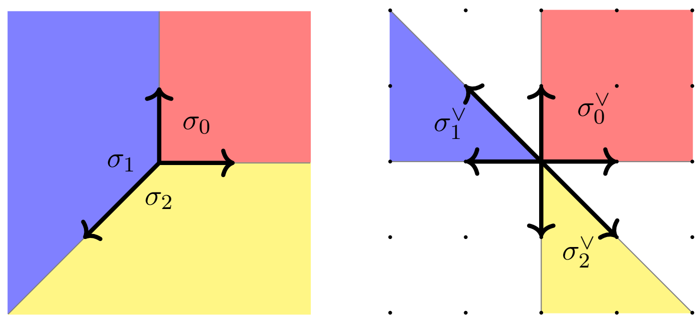

Figure 1: The fan $\Sigma_{\mathbb{P}^2}$ (left) and the duals of its maximal cones (right).

In particular, the generators of $\sigma_1^\vee \cap M$ *and of* $\sigma_2^\vee \cap M$ *collectively generate* $\tau^\vee \cap M$*.*

The separation lemma implies that we need only consider gluing the U_σ for maximal cones $\sigma \in \Sigma_{\mathbb{P}^2}$, and the rest follows automatically. Together with Proposition 2.7, we see that all the information we need to construct $\mathbb{P}^2$ is contained within the generators of the $\sigma_i^\vee$.

Example 2.9 ([CLS, Example 3.1.9])**.** Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$. We have

$$\begin{aligned}\sigma_0^\vee &= \operatorname{Cone}(e_1, e_2),\\ \sigma_1^\vee &= \operatorname{Cone}(-e_1, -e_1 + e_2),\\ \sigma_2^\vee &= \operatorname{Cone}(e_1 - e_2, -e_2).\end{aligned}$$

Thinking of M as a lattice of monomials $\chi^m = x^{m_1} y^{m_2}$, these cones give us the affine toric varieties

$$\begin{aligned}U_{\sigma_0} &= \operatorname{Spec} \mathbb{C}[x, y],\\ U_{\sigma_1} &= \operatorname{Spec} \mathbb{C}[x^{-1}, x^{-1}y],\\ U_{\sigma_2} &= \operatorname{Spec} \mathbb{C}[xy^{-1}, y^{-1}],\end{aligned}$$

by Proposition 2.7. All three of these varieties are isomorphic to $\mathbb{C}^2$, but they have different local coordinates, determined by the monomials generating the coordinate rings.

The common faces of each pair of maximal cones determine how we glue them

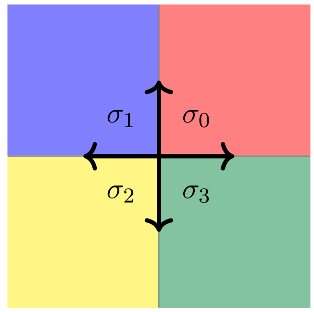

Figure 2: The fan of $\mathbb{P}^1 \times \mathbb{P}^1$.

together, and correspond to the following toric varieties by Lemma 2.8.

$$\begin{aligned}
U_{\sigma_0 \cap \sigma_1} &= \operatorname{Spec} \mathbb{C}[x, y, x^{-1}, x^{-1}y] \\
&= \operatorname{Spec} \mathbb{C}[x, x^{-1}, y] = \mathbb{C}^* \times \mathbb{C}, \\
U_{\sigma_0 \cap \sigma_2} &= \operatorname{Spec} \mathbb{C}[x, y, xy^{-1}, y^{-1}] \\
&= \operatorname{Spec} \mathbb{C}[x, y, y^{-1}] = \mathbb{C} \times \mathbb{C}^*, \\
U_{\sigma_1 \cap \sigma_2} &= \operatorname{Spec} \mathbb{C}[x^{-1}, x^{-1}y, xy^{-1}, y^{-1}] \\
&= \operatorname{Spec} \mathbb{C}[x^{-1}, x^{-1}y, xy^{-1}] = \mathbb{C} \times \mathbb{C}^*.
\end{aligned}$$

These are the shared open subsets along which we can glue the U_{σ_i}.

Equip $\mathbb{P}^2$ with homogeneous coordinates $[t_0 : t_1 : t_2]$ and, for each $i \in \{0, 1, 2\}$, let U_i be the open set on which $t_i \neq 0$. Then the morphism $x \mapsto t_1/t_0, y \mapsto t_2/t_0$ identifies each U_{σ_i} with U_i. So $X_{\Sigma_{\mathbb{P}^2}} = \mathbb{P}^2$ as expected. $\Diamond$

Example 2.10. Another example of a toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$. Figure 2 shows its fan. We omit the details of the gluing here; it will be treated in greater generality in Example 4.8. $\Diamond$

2.4 Toric Varieties from Polytopes

Toric varieties can also be constructed from convex polytopes. This can be done directly with semigroups, but here we only show how to derive fans from lattice polytopes.

Definition 2.11. A *lattice polytope* in $M_{\mathbb{R}}$ is a set

$$P = \operatorname{Conv}(u_1, \dots, u_k) = \left\{ \lambda_1 u_1 + \cdots + \lambda_n u_n \mid \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}$$

for some finite set $\{u_1, \dots, u_n\} \in M$. We call P the *convex hull* of $\{u_1, \dots, u_n\}$.

The *dimension* of a lattice polytope P is the dimension of the smallest affine subspace containing P. A subset $Q \subseteq P$ is a *face* of P if there is a nonzero vector $u \in N_{\mathbb{R}}$ and a real number b such that the following hold:

(i) P lies entirely within the halfspace $\{m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq b\}$.

(ii) Q is the intersection of P with the affine hyperplane $\{m \in M_{\mathbb{R}} \mid \langle m, u \rangle = b\}$.

The dimension of Q is defined in the same way as that of P – we call a face of dimension $(\dim P - 1)$ a *facet* and a face of dimension 1 a *vertex*. We denote by $V(P)$ the set of vertices of P. Any polytope is the convex hull of its own vertices.

From here on we consider only full-dimensional lattice polytopes. If P is such a polytope, and F is a facet of P, then the inward-pointing normal vectors of F lie on a ray in $N_{\mathbb{R}}$. We can thus pick a minimal generator u_F of this ray. The value $\langle v, u_F \rangle$ is constant as v varies over the vertices of F, and is an integer – denote this value by $-a_F$. We then have another description of P, this time as an intersection of halfspaces:

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \text{ of } P\}. \tag{3}$$

We call (3) the *facet presentation* of P, and this is unique.

We want to use a lattice polytope P to construct a fan, from which we can then construct a toric variety. To ensure that this fan is regular, we require for the rest of this subsection that the vertices of each facet of P form a $\mathbb{Z}$-basis of M.

Definition 2.12. Let $P \subseteq M_{\mathbb{R}}$ be a lattice polytope. For any face Q of P we can define the cone

$$\sigma_Q = \mathrm{Cone}(\{u_F \mid F \text{ is a facet and } Q \subseteq F\}) \subseteq N_{\mathbb{R}}.$$

In particular, if $v \in V(P)$ is a vertex and $F_1, \ldots F_k$ are the facets of P containing v, we have the cone

$$\sigma_v = \mathrm{Cone}(u_{F_1}, \ldots, u_{F_k}) \subseteq N_{\mathbb{R}}.$$

The collection

$$\Sigma = \{\sigma_Q \mid Q \text{ is a face of } P\}$$

is a fan, called the *normal fan* of P.

In a certain special case, there is another (equivalent) construction.

Definition 2.13. A lattice polytope P is *reflexive* if its facet presentation is

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -1 \text{ for all facets } F \text{ of } P\}.$$

The only lattice point in the interior of a reflexive polytope is the origin.

If $P \subseteq M_{\mathbb{R}}$ is reflexive, then its *dual polytope*

$$P^\vee = \{u \in N_{\mathbb{R}} \mid \langle m, u\rangle \geq -1 \text{ for all } m \in P\} \subseteq N_{\mathbb{R}}$$

is of the form $P^\vee = \mathrm{Conv}(\{u_F \mid F \text{ is a facet of } P\})$, and is thus a lattice polytope. Each face Q of $P^\vee$ then defines the cone $\mathrm{Cone}(V(Q))$ whose generators are the vertices of Q. These cones form a fan, which we call the fan of *cones over the faces of* $P^\vee$. It is clear that this fan coincides with the normal fan of P.

Reflexive polytopes will be useful in Section 5. Indeed, Figure 5 shows a reflexive polytope and its dual – the normal fan of the polytope on the left is the fan over the faces of the one on the right, and this is the fan of $\mathbb{P}^2$.

2.5 Orbits and Divisors

An important part of the definition of a toric variety X_Σ is the torus action defined on it. Looking at the cones of Σ tells us a lot about the orbits of this action.

The characterisation of the torus orbits that we will use relies on a bijective correspondence unique to toric varieties: every point of an affine toric variety U_σ corresponds to a semigroup homomorphism $\mathsf{S} = \sigma^\vee \cap M \to \mathbb{C}$ (where $\mathbb{C}$ is considered as a semigroup under multiplication) and vice versa. Given a point $p \in U_\sigma$, the corresponding semigroup homomorphism is the map taking $m \in \mathsf{S}$ to $\chi^m(p) \in \mathbb{C}$, and every homomorphism $\mathsf{S} \to \mathbb{C}$ arises this way (see [CLS, Proposition 1.3.1]).

Let σ be a cone. We define $\sigma^\perp \subseteq \sigma^\vee$ by replacing the inequality in (2) with an equality. Then we can define the *distinguished point* of the affine toric variety U_σ, denoted γ_σ, which is the point corresponding to the semigroup homomorphism $\sigma^\vee \cap M \to \mathbb{C}$ given by

$$u \mapsto \begin{cases} 1 \text{ if } u \in \sigma^\perp, \\ 0 \text{ otherwise.} \end{cases}$$

Importantly, for two cones σ, σ', we have $\gamma_\sigma = \gamma_{\sigma'}$ if and only if $\sigma = \sigma'$.

Given a toric variety X_Σ with torus $T = (\mathbb{C}^*)^n$, the T-orbits in X_Σ are exactly the sets

$$O(\sigma) = T \cdot \gamma_\sigma \cong (\mathbb{C}^*)^{n - \dim \sigma}$$

as σ varies over Σ. In particular, each cone corresponds to a unique T-orbit.

The closure of an orbit, $\overline{O(\sigma)}$, is a toric variety with torus $O(\sigma)$ and dimension $n - \dim \sigma$. In particular, the rays of Σ give rise to toric subvarieties of dimension $n-1$. Such subvarieties are important objects of study in algebraic geometry.

A *prime divisor* of a variety X is a subvariety of codimension 1. A *Weil divisor* is an element of the free abelian group $\mathrm{Div}(X)$ generated by the prime divisors of X, i.e. a finite formal sum $\sum a_i D_i$, where the a_i are integers and the D_i are prime divisors. We say D is *effective* if all the a_i are non-negative. We can define the restriction of a Weil divisor $D = \sum a_i D_i$ to an open subset $U \subseteq X$ by

$$D|_U = \sum_{D_i \cap U \neq \varnothing} a_i(D_i \cap U).$$

Two other important types of (Weil) divisor are *principal divisors*, which arise from functions, and *Cartier divisors*, which are locally principal, i.e. restrict to (the restriction of) some principal divisor on each set of some open cover. The principal divisor of $f \in \mathbb{C}(X)^*$ is a formal sum over all principal divisors,

$$\operatorname{div}(f) = \sum_D \nu_D(f) D.$$

Here $\mathbb{C}(X)$ is the field of rational functions defined on X – see [CLS, §3.0] for details – which in the affine case is the field of fractions of the coordinate ring, and $\nu_D(f)$ is the order of vanishing of f on D. The order of vanishing of any such f is in fact 0 for all but a finite number of prime divisors, so $\operatorname{div}(f)$ is indeed a Weil divisor. We denote by $\operatorname{Div}_0(X)$ and $\operatorname{CDiv}(X)$ respectively the groups of principal and Cartier divisors on X, and note that principal divisors are clearly Cartier.

It is worth noting that on a smooth variety, all Weil divisors are Cartier. From here on we will therefore favour the terminology of Cartier divisors, even when a result may apply to Weil divisors in the general case.

Two Cartier divisors are called *linearly equivalent* if they differ by a principal divisor. As $\operatorname{Div}_0(X)$ is a subgroup of $\operatorname{CDiv}(X)$, we can define the *Picard group* of linear equivalence classes by

$$\operatorname{Pic}(X) = \operatorname{CDiv}(X)/\operatorname{Div}_0(X).$$

There is an alternative way of defining the Picard group which is special to the toric case and which will be more useful to us. To state it we need to return to the orbit closures described above.

Let Σ be a fan and denote by $\Sigma(1)$ the set of rays of this fan. For each ray $\rho \in \Sigma(1)$, the orbit closure $D_\rho = \overline{O(\rho)}$ is a prime divisor on X_Σ, and as the torus action on X_Σ sends D_ρ to itself it is a *torus-invariant* prime divisor. Taking formal sums of these gives torus-invariant Cartier divisors on X_Σ, which form a group denoted by $\operatorname{CDiv}_T(X_\Sigma)$.

Each element of the dual lattice M has a corresponding torus-invariant Cartier divisor. Let $u_\rho \in G(\Sigma)$ denote the generator of the ray ρ for each $\rho \in \Sigma(1)$, and let $\chi^m \in M$. Then we define[3]

$$\operatorname{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

[3]In fact, if we consider χ^m as a function on X, then this is just a special case of the definition of a principal divisor.

This allows us to see M as a subgroup of $\mathrm{CDiv}_T(X_\Sigma)$, and gives us the following definition.

Definition 2.14. The Picard group of an algebraic variety X can be defined as the quotient

$$\mathrm{Pic}(X) = \mathrm{CDiv}_T(X)/M.$$

Proof of equivalence of the two definitions can be found in [CLS, §4.2]. This definition has two interesting consequences:

- If all we care about are properties of divisors up to linear equivalence, then we can restrict our focus to only torus-invariant divisors.
- The *Picard number* of X_Σ, i.e. the rank of $\mathrm{Pic}(X_\Sigma)$, is precisely $|G(\Sigma)| - \dim \Sigma$.

2.6 $\mathcal{O}_X(D)$ and $\mathcal{O}_{\mathbb{P}^k}(1)$

Sheaves are important structures for the study of algebraic geometry, encoding information about varieties (or more generally schemes) on a local level. We omit the formal definition here – see [Har, Ch. II §1] for details – but it is useful to know the following:

- A sheaf on a variety X associates an object (in our case often a ring or a module) with each open subset of X.
- To specify a sheaf it suffices to specify its behaviour on an arbitrary open subset of X.

An important example for algebraic geometry is the *structure sheaf* of a variety X, denoted $\mathcal{O}_X$, in which the structure corresponding to an open set $U \subseteq X$ is the ring of regular functions on U. In particular when X is affine, we have $\mathcal{O}_X(X) = \mathbb{C}[X]$, the coordinate ring. For any open subset $U \subseteq X$, we have

$$\mathcal{O}_X(U) = \{f \in \mathbb{C}(X)^* \mid \mathrm{div}(f)|_U \geq 0\} \cup \{0\}.$$

(If we want to restrict ourselves to an open subset V of X, we simply define the restriction of a sheaf $\mathcal{F}$ by $(\mathcal{F})|_V(U) = \mathcal{F}(U)$ for all open $U \subseteq V$.)

Similarly, any Cartier divisor D gives a sheaf $\mathcal{O}_X(D)$ as follows:

$$\mathcal{O}_X(D)(U) = \{f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}.$$

This is not a ring, but an $\mathcal{O}_X(U)$-module – we say $\mathcal{O}_X(D)$ is a *sheaf of $\mathcal{O}_X$-modules*. When D is effective, we can think of this as giving the regular functions on U as well as those with poles that are "only so bad" – i.e. the orders of the poles of f along prime divisors intersecting with U are limited by the corresponding coefficients in the sum defining D. In particular, if $D = 0 \in \mathrm{CDiv}(X)$, then $\mathcal{O}_X(D) = \mathcal{O}_X$.

An important fact is that linearly equivalent divisors give rise to isomorphic sheaves. This leads to the idea of defining $\mathcal{O}_X(D)$ using the equivalence class of D

in Pic(X) rather than the divisor itself. The following example applies this idea to projective space.

Example 2.15 ([CLS, Examples 4.1.6 and 4.3.1])**.** In $\mathbb{P}^k$ there are $k+1$ torus-invariant prime divisors $D_0, \ldots, D_k$ corresponding to the generators $\ell_0, \ldots, \ell_k$ defined in Section 2.3.

We compute the Picard group of $\mathbb{P}^k$ using Definition 2.14. The map $M \to \mathrm{CDiv}_T(\mathbb{P}^k)$ given by

$$\begin{gathered} \mathbb{Z}^k \to \mathbb{Z}^{k+1} \\ (a_1, \ldots, a_k) \mapsto (-a_1 - \cdots - a_k, a_1, \ldots, a_k), \end{gathered}$$

embeds M as a subgroup of $\mathrm{CDiv}_T(\mathbb{P}^k)$. Then the quotient map

$$\begin{gathered} \mathbb{Z}^{k+1} \cong \mathrm{CDiv}_T(\mathbb{P}^k) \to \mathrm{CDiv}_T(\mathbb{P}^k)/M \cong \mathbb{Z} \\ (a_0, \ldots, a_k) \mapsto a_0 + \cdots + a_k, \end{gathered}$$

gives us $\mathrm{Pic}(\mathbb{P}^k) \cong \mathbb{Z}$.

In particular, $D_0, \ldots, D_k$ all map to $1 \in \mathrm{Pic}(\mathbb{P}^k)$, so they are linearly equivalent. Hence the corresponding sheaves $\mathscr{O}_{\mathbb{P}^k}(D_i)$ are isomorphic. We denote these sheaves by $\mathscr{O}_{\mathbb{P}^k}(1)$ and, more generally, for any divisor D on $\mathbb{P}^k$ with equivalence class $a \in \mathbb{Z} \cong \mathrm{Pic}(\mathbb{P}^k)$ we denote $\mathscr{O}_{\mathbb{P}^k}(D)$ by $\mathscr{O}_{\mathbb{P}^k}(a)$. ♢

3 Primitive Collections and Primitive Relations

In this section we introduce the basic concepts from [Bat1], which we use heavily in the following sections. The key feature of Batyrev's paper is the concept of *primitive collections*, which are simple combinatorial objects associated with a fan that nonetheless manage to capture some of the behaviour of the associated toric variety.

Definition 3.1. Let Σ be a fan. A non-empty subset $\mathscr{P} = \{u_1, \ldots, u_k\} \subseteq G(\Sigma)$ is a *primitive collection* if the elements of $\mathscr{P}$ do not generate any cone in Σ, but for any $i \in \{1, \ldots, k\}$, the elements of $\mathscr{P} \setminus \{u_i\}$ do generate a cone in Σ.

The *focus* $\sigma(\mathscr{P})$ of a primitive collection $\mathscr{P}$ as above is defined to be the smallest cone (by dimension) of Σ containing $u_1 + \cdots + u_k$. Such a cone exists because the fan Σ is complete.

Key to the use of primitive collections are the corresponding *primitive relations*.

Definition 3.2. Let $\mathscr{P} = \{u_1, \ldots, u_k\}$ be a primitive collection with focus $\sigma(\mathscr{P}) = \mathrm{Cone}(v_1, \ldots, v_m)$. Then there is a unique set of positive integer coefficients a_i such that

$$u_1 + \cdots + u_k - a_1 v_1 - \cdots - a_m v_m = 0. \tag{4}$$

We call (4) the *primitive relation* associated with $\mathscr{P}$, and we call the integer $D(\mathscr{P}) = k - \sum_{i=1}^{m} a_i$ the *degree* of the collection.

We now establish some key facts about the above objects.

Proposition 3.3. *Let Σ be a fan. Every subset of $G(\Sigma)$ not generating a cone of Σ includes a primitive collection. In particular, every fan has at least one primitive collection.*

The proof of the first statement is straightforward, using induction. The second statement then follows immediately from the first, as we require fans to be complete.

Proposition 3.4. *Let $\mathscr{P} \subseteq G(\Sigma)$ be a primitive collection with focus $\sigma(\mathscr{P})$. Then $\mathscr{P} \cap \sigma(\mathscr{P}) = \varnothing$.*

Proof. Let $\mathscr{P} = \{u_1, \dots, u_k\}$ and let $v_1, \dots, v_m \in G(\Sigma)$ be the generators of $\sigma(\mathscr{P})$. As $\mathscr{P} \subseteq G(\Sigma)$, it is sufficient to prove that $\mathscr{P} \cap \{v_1, \dots, v_m\} = \varnothing$.

Suppose $u_1 = v_1$. Then we have $u = u_2 + \cdots + u_k \in \tau_1 = \operatorname{Cone}(u_2, \dots, u_k)$, and τ_1 is a cone in Σ as $\mathscr{P}$ is a primitive collection. However, applying $u_1 = v_1$ to the primitive relation associated with $\mathscr{P}$ gives

$$u = (a_1 - 1)v_1 + a_2 v_2 + \cdots + a_m v_m$$

so $u \in \tau_2 \in \Sigma$ where $\tau_2 = \operatorname{Cone}(v_1, \dots, v_m)$ if $a_1 \geq 2$, or $\tau_2 = \operatorname{Cone}(v_2, \dots, v_m)$ if $a_1 = 1$. Hence $\tau_1 \cap \tau_2$ contains u and is a face of both τ_1 and τ_2 by the definition of a fan. As $\{u_2, \dots, u_k\}$ is linearly independent, τ_1 is the smallest cone containing u, so $\tau_1 = \tau_1 \cap \tau_2 \preceq \tau_2$. Similarly, $\{v_1, \dots, v_m\}$ is linearly independent, so τ_2 is the smallest cone containing u, and $\tau_2 \preceq \tau_1$. Thus $\tau_1 = \tau_2$, which is only possible if the generating sets of τ_1 and τ_2 are equal.

If $a_1 = 1$, then $\mathscr{P} = \{v_1, \dots, v_m\}$, which is impossible as the left hand side is a primitive collection and the right hand side generates $\sigma(\mathscr{P})$. The only other case is $a_1 = 2$, in which case $v_1 = u_1$ is a generator of τ_2 and hence coincides with one of $u_2, \dots, u_k$, which is impossible as the u_i must all be distinct. We have a contradiction. □

Proposition 3.5. *Let Σ be a fan in $\mathbb{R}^n$. Then each generator $u \in G(\Sigma)$ defines a fan Σ_u in $\mathbb{R}^{n-1}$ by taking the cones of Σ_u to be precisely the images of the cones of Σ containing u by projection onto $\mathbb{R}^n / \operatorname{span}\{u\}$. The fan generators of Σ_u are then precisely the images of those $v \in G(\Sigma) \setminus \{u\}$ such that $\operatorname{Cone}(u, v) \in \Sigma$.*

The fan Σ_u defines a toric subvariety of X_Σ of codimension 1, i.e. a torus-invariant divisor of X_Σ. This variety is precisely the orbit closure $\overline{O(\rho_u)}$ which we met in Section 2.5.

Proof of Proposition 3.5. Fix $u \in G(\Sigma)$ and let $p : \mathbb{R}^n \to \mathbb{R}^n / \operatorname{span}\{u\}$ be the projection map. This map is linear, so Σ_u inherits the fan properties from Σ, but not necessarily completeness. If Σ_u is not complete, then there exists $\bar{v} \in \mathbb{R}^n / \operatorname{span}\{u\}$ not lying in any cone of Σ_u. Hence no $v \in p^{-1}(\bar{v})$ lies in a cone of Σ containing u, which is only possible if Σ is not complete. Thus Σ_u must be complete.

If $\operatorname{Cone}(u, v) \in \Sigma$ then $\operatorname{Cone}(p(v)) \in \Sigma_u$ by definition, so $p(v) \in G(\Sigma_u)$. Conversely, if a lattice point $\bar{v} \in \mathbb{Z}^n / \mathbb{Z}\{u\}$ is in $G(\Sigma_u)$, then it has minimal norm and $\operatorname{Cone}(\bar{v}) = p(\sigma)$ for some cone $\sigma \in \Sigma$. In this case, there exists $v \in p^{-1}(\bar{v}) \cap G(\Sigma)$ which is a generator of σ, and by definition of Σ_u we also have that u is a generator of σ, so $\operatorname{Cone}(u, v) \preceq \sigma$ and thus $\operatorname{Cone}(u, v) \in \Sigma$. □

The primitive collections of Σ_u are closely related to those of Σ.

Proposition 3.6. *Let $\{\bar{u}_1, \ldots, \bar{u}_k\} \in G(\Sigma_u)$ be a primitive collection. Then*

$$\{u, u_1, \ldots, u_k\} \text{ or } \{u_1, \ldots, u_k\}$$

is a primitive collection in $G(\Sigma)$.

Proof. By Proposition 3.5, $S = \{u, u_1, \ldots, u_k\}$ does not generate any cone in Σ, else $\{\bar{u}_1, \ldots, \bar{u}_k\}$ would generate a cone in Σ_u. Hence there is a primitive collection $\mathscr{P} \subseteq S$ by Proposition 3.3. For any $i \in \{1, \ldots, k\}$, we have that $\{\bar{u}_1, \ldots, \bar{u}_k\} \setminus \{\bar{u}_i\}$ generates some cone in Σ_u, so $S \setminus \{u_i\}$ generates some cone in Σ. Hence $\{u_1, \ldots, u_k\} \subseteq \mathscr{P}$, which gives the stated result. □

4 When is a Toric Variety a Projective Bundle?

In [Bat1], Batyrev uses primitive collections and primitive relations to characterise when a toric variety is a projective bundle. To understand what this means, we must first define a more general structure.

Definition 4.1. Let E, F, and B be algebraic varieties. We say that E is an F-bundle over B if there is an open cover $\mathcal{U}$ of B and a surjective morphism $p : E \to B$ such that for each open set $U \in \mathcal{U}$ there is an isomorphism $\varphi : p^{-1}(U) \to U \times F$ which makes the following diagram commute.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow[\cong]{\varphi} & U \times F \\ & \searrow^{p} & \downarrow{\scriptstyle \mathrm{pr}_1} \\ & & U \end{array}$$

(Where pr_1 is projection onto the first factor.) We call F and B the *fibre* and the *base space* respectively; we call the pairs (U, φ) *trivialisations*, and E (or p) a *fibre bundle*.

In other words, a fibre bundle looks locally like a product space. In the notation of Definition 4.1, the variety E looks locally like $B \times F$. If we want a toric fibre bundle, we take E, B, and F to be toric varieties, and (to preserve the toric structure) we require p and every trivialisation isomorphism φ to be equivariant with respect to the torus action.

Note that for any two distinct trivialisations (U_i, φ_i) and (U_j, φ_j) with $U_i \cap U_j$ nonempty, there are two different ways of viewing $p^{-1}(U_i \cap U_j)$ as the product $(U_i \cap U_j) \times F$: the "U_i point of view" and the "U_j point of view". The natural way to get from the former to the latter is using the automorphism

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

which we temporarily denote by $\varphi_j \circ \varphi_i^{-1} = (\psi_1, \psi_2)$.[4] We then have a map

$$\begin{aligned} g_{ij} : U_i \cap U_j &\to \mathrm{Aut}(F) \\ u &\mapsto \psi_2(u, \cdot) \end{aligned}$$

which we call a *transition function* – these will become important later. The idea is that for any $u \in U_i \cap U_j$, the function $g_{ij}(u)$ tells us how to convert between the different ways of seeing the fibre $p^{-1}(u) \cong F$.

For now, we'd like to understand when one toric variety is a toric fibre bundle over another. Proposition 4.3 gives criteria for this, but to understand the statement we first need to understand what a "structure-preserving map" looks like for fans, and how it corresponds to morphisms of varieties.

Definition 4.2 (Adapted from [Oda]). Let N, N' be lattices and let $\Sigma \subseteq N_{\mathbb{R}}, \Sigma' \subseteq N'_{\mathbb{R}}$ be fans. A linear map $h : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ is a *map of fans* if it satisfies the following conditions:

(i) $h(N) \subseteq N'$. (Note that $h|_N$ is always a group homomorphism.)

(ii) $N'/h(N)$ is finite.

(iii) For any $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma'$ such that $h(\sigma) \subseteq \sigma'$.

In this case we write $h : \Sigma \to \Sigma'$.

A map of fans induces a morphism between the corresponding toric varieties. This morphism is equivariant as h maps cones to cones, and is *dominant* (i.e. has dense image) as $0 \in N'_{\mathbb{R}}$ is always in the image of h, and as $U_{\{0\}}$ is the torus of $X_{\Sigma'}$ and thus dense.

Moreover, a map of fans induces linear maps between dual cones in a manner analogous to the construction of a homomorphism $\mathbb{C}[Y] \to \mathbb{C}[X]$ from a morphism

[4]Note that we must have $u = \mathrm{pr}_1(u, v) = \mathrm{pr}_1(\varphi_j \circ \varphi_i^{-1}(u, v))$, so in fact $\psi_1(u, v) = u$ for all $(u, v) \in U_i \cap U_j$. That is, the automorphism $\varphi_j \circ \varphi_i^{-1}$ fixes $U_i \cap U_j$, and so transition functions really are all we need to switch point of view.

of varieties $X \to Y$. We consider elements of M' as linear maps $N' \to \mathbb{Z}$ (and, by extension, elements of $M'_{\mathbb{R}}$ as linear maps $N'_{\mathbb{R}} \to \mathbb{R}$). Consider cones $\sigma \in \Sigma, \sigma' \in \Sigma'$ such that $h(\sigma) \subseteq \sigma'$. Then the restriction $h|_{\sigma} : \sigma \to \sigma'$ induces a map

$$\psi : (\sigma')^{\vee} \to \sigma^{\vee}, \ u \mapsto u \circ h.$$

Let $u \in (\sigma')^{\vee}$. By definition, we have $u(v') \geq 0$ for all $v' \in \sigma'$. We also have $h(v) \in \sigma'$ for any $v \in \sigma$, so in particular $u \circ h(v) \geq 0$. Hence $u \circ h \in \sigma^{\vee}$, so ψ is well-defined.

In particular, if $h|_{\sigma}$ as above is surjective, then the only $u \in (\sigma')^{\vee}$ satisfying $u \circ h = 0$ is $u = 0$. Hence $\ker \psi = \{0\}$ and thus h induces an injective map between dual cones.

It is straightforward to see how a linear map between dual cones gives rise to a map of semigroups, and then a morphism of $\mathbb{C}$-algebras, using the constructions given in Section 2.2. Additionally, this process is fully reversible, so a homomorphism between coordinate rings $\mathbb{C}[U_{\sigma'}] \to \mathbb{C}[U_{\sigma}]$ gives us a map $\sigma \to \sigma'$. These can be patched together to get a map of fans. The correspondence

$$\{\text{Morphisms of affine varieties } X \to Y\} \quad \longleftrightarrow \quad \{\text{Homomorphisms } \mathbb{C}[Y] \to \mathbb{C}[X]\}$$

is a standard fact of algebraic geometry, and a surjective morphism of affine varieties corresponds to an injective $\mathbb{C}$-algebra homomorphism.

We can now state the result about toric fibre bundles.

Proposition 4.3 ([Oda, Proposition 7.3])**.** *Let Σ, Σ_B be complete fans in $N_{\mathbb{R}}, (N_B)_{\mathbb{R}}$ respectively, with corresponding varieties $X = X_{\Sigma},\ B = X_{\Sigma_B}$. Let $h : \Sigma \to \Sigma_B$ be a map of fans and let $f : X \to B$ be the corresponding morphism of varieties. Let N_F be the kernel of $h|_N : N \to N_B$ and consider a fan Σ_F in $(N_F)_{\mathbb{R}}$ with corresponding variety $F = X_{\Sigma_F}$. Then f is a toric fibre bundle if and only if the following conditions hold:*

- *(O1) The group homomorphism $h|_N : N \to N_B$ is surjective.*
- *(O2) The fan Σ_B has a lift $\tilde{\Sigma}_B \subseteq \Sigma$. That is, there is a (not necessarily complete) fan $\tilde{\Sigma}_B \subseteq \Sigma$ in $N_{\mathbb{R}}$ such that for each $\sigma_B \in \Sigma_B$ there is a unique $\tilde{\sigma}_B \in \tilde{\Sigma}_B$ such that h restricts to a bijection $\tilde{\sigma}_B \to \sigma_B$.*
- *(O3) The cones of Σ are exactly those of the form*

$$\sigma = \tilde{\sigma}_B + \sigma_F$$

where $\tilde{\sigma}_B \in \tilde{\Sigma}_B$ and $\sigma_F \in \Sigma_F$.

Proof. We first treat the "if" implication. The first condition implies that f is a surjection. To see this, consider a cone $\sigma_B \in \Sigma_B$. As h is a surjective map of fans, the cone σ_B is the image of at least one cone $\sigma \in \Sigma$. The restriction of h to a map $\sigma \to \sigma_B$ induces a surjective morphism of affine varieties $U_\sigma \to U_{\sigma_B}$ as discussed above. As the U_{σ_B} cover B, we get a surjection $f : X \to B$.

Note that $F = \ker f$, and that $\{U_{\sigma_B} \mid \sigma_B \in \Sigma_B\}$ is an open cover of B. Condition (O2) gives us a unique cone $\tilde{\sigma}_B \in \Sigma$ such that $f|_{U_{\tilde{\sigma}_B}} : U_{\tilde{\sigma}_B} \to U_{\sigma_B}$ is a bijection (by a similar argument to the one above) and hence an isomorphism since the codomain is normal.

By (O3) we have $f^{-1}(U_{\sigma_B}) = U_{\tilde{\sigma}_B} \times F$ and we can define

$$\begin{aligned} \varphi : U_{\tilde{\sigma}_B} \times F &\to U_{\sigma_B} \times F, \\ (u, v) &\mapsto (f|_{U_{\tilde{\sigma}_B}}(u), v), \end{aligned}$$

which is clearly an equivariant isomorphism. By construction, $\mathrm{pr}_1 \circ \varphi$ coincides with f on $f^{-1}(U_{\sigma_B})$. Thus f is a toric fibre bundle.

We now treat the "only if" implication. Suppose f is an equivariant fibre bundle, with $\dim X_\Sigma = d, \dim F = k$. The condition (O1) is immediate from the surjectivity of f.

By equivariance of f, we can assume the open cover $\mathcal{U}$ from Definition 4.1 is $\{U_{\sigma_B} \mid \sigma_B \in \Sigma_B\}$. We obtain (O2) by lifting each U_{σ_B} to $\varphi^{-1}(U_{\sigma_B} \times (\mathbb{C}^*)^k)$. Again by equivariance, this is an open affine toric variety of $f^{-1}(U_{\sigma_B})$, hence associated with some cone $\tilde{\sigma}_B \in \Sigma$. We define $\tilde{\Sigma}_B$ to be the collection of these lifted cones and notice that it inherits the fan properties (though not completeness) from Σ.

For (O3), first notice that we must have $\Sigma_F \subseteq \Sigma$ (after inclusion $(N_F)_\mathbb{R} \hookrightarrow N_\mathbb{R}$). This is because f is trivial over the torus of B, that is $f^{-1}((\mathbb{C}^*)^{d-k}) = (\mathbb{C}^*)^{d-k} \times F$, and so $h^{-1}(\{0\}) = \{0\} \times \Sigma_F$. Now let $\sigma_B \in \Sigma_B, \sigma_F \in \Sigma_F$ and consider $U_{\sigma_B} \times U_{\sigma_F}$. This is an open affine toric subvariety of $U_{\sigma_B} \times F$ and so its preimage by φ is an open affine toric subvariety of $f^{-1}(U_{\sigma_B})$ associated with some cone $\sigma \in \Sigma$. We must then have

$$\sigma = \tilde{\sigma}_B + \sigma_F.$$

By equivariance, all open toric subvarieties of X arise this way, so all cones in Σ are of this form. □

We can use this result to characterise projective bundles using primitive collections and primitive relations.

Theorem 4.4 ([Bat1, Proposition 4.1])**.** *Let Σ be a d-dimensional fan. The corresponding toric variety X_Σ is a toric $\mathbb{P}^k$-bundle over a $(d-k)$-dimensional toric variety B if and only if there is a primitive collection $\mathscr{P} = \{u_0, \ldots, u_k\} \subseteq G(\Sigma)$ satisfying the following conditions:*

(i) The primitive relation associated with $\mathscr{P}$ is

$$u_0 + \cdots + u_k = 0.$$

(ii) For any other primitive collection $\mathscr{P}' \subseteq G(\Sigma)$ we have $\mathscr{P} \cap \mathscr{P}' = \varnothing$.

Proof. We first treat the "only if" case. We have a surjective morphism $f : X_\Sigma \to B$ which induces a surjective map of fans $h : \Sigma \to \Sigma_B$. As X_Σ is a $\mathbb{P}^k$-bundle, we have $F = \mathbb{P}^k$ and so Σ_F is as in Section 2.3. In particular, $\mathscr{P} = G(\Sigma_F) \subseteq G(\Sigma)$ is a primitive collection with the corresponding primitive relation being that of (i).

Let $\mathscr{P}' \subseteq G(\Sigma)$ be a primitive collection not equal to $\mathscr{P}$, and suppose for contradiction that there exists $u^* \in \mathscr{P} \cap \mathscr{P}'$. Then $\mathrm{Cone}(u^*) \in \Sigma$ and $\mathrm{Cone}(\mathscr{P}' \setminus \{u^*\}) \in \Sigma$. Moreover, $\mathrm{Cone}(u^*) \in \Sigma_F$ and by (O3) we can write $\mathrm{Cone}(\mathscr{P}' \setminus \{u^*\}) = \tilde{\sigma}_B + \sigma_F$.

As $\mathscr{P} \neq \mathscr{P}'$, we must have $\mathscr{P} \nsubseteq \mathscr{P}'$ as both are primitive collections. Let $v^* \in \mathscr{P} \setminus \mathscr{P}'$. Then the generators of σ_F form a subset of $\mathscr{P} \setminus \{u^*, v^*\}$, so the cone $\sigma_F^* = \sigma_F + \mathrm{Cone}(u^*)$ is in Σ_F. Hence

$$\mathrm{Cone}(\mathscr{P}' \setminus \{u^*\}) + \mathrm{Cone}(u^*) = \tilde{\sigma}_B + \sigma_F^* \in \Sigma$$

again by (O3), but this is impossible as the right hand side is $\mathrm{Cone}(\mathscr{P}')$, and $\mathscr{P}'$ is a primitive collection. Condition (ii) is therefore satisfied.

We now prove the "if" case, breaking it up into steps.

Step 1. As $\mathscr{P}$ is a primitive collection, $\mathscr{A} = \{u_1, \ldots, u_k\}$ generates a cone in Σ. As Σ is regular, $\mathrm{Cone}(\mathscr{A})$ is regular, so $\mathscr{A}$ is a subset of a basis for N. Let $N_F = \mathbb{Z}\mathscr{A}$, $N_B = N/N_F$, and let h be the quotient map

$$h : N_\mathbb{R} \to (N_B)_\mathbb{R} = N_\mathbb{R}/(N_F)_\mathbb{R}.$$

Then $h|_N : N \to N_B$ is a quotient map of groups, hence a surjective group homomorphism. Thus (O1) holds.

Step 2. We now work towards proving both (O2) and (O3). Consider a set $\{v_0, \ldots, v_s\} \subseteq G(\Sigma) \setminus \mathscr{P}$ generating a cone in Σ. Arguing by induction on s, condition (ii) implies that for any $i \in \{0, \ldots, k\}$, the set $\{u_i, v_0, \ldots, v_s\}$ also generates a cone in Σ.

Now consider $\{u_{i_0}, \ldots u_{i_r}\} \subseteq \mathscr{P}$ and suppose for induction that if $\{v_0, \ldots, v_s\} \in G(\Sigma) \setminus \mathscr{P}$ generates a cone in Σ then $\{u_{i_0}, \ldots u_{i_{r-1}}, v_0, \ldots, v_s\}$ also generates a cone in Σ. Then

$$S = \{u_{i_0}, \ldots u_{i_r}, v_0, \ldots, v_s\}$$

is not primitive by (ii), so either $\mathrm{Cone}(S) \in \Sigma$ or there exists some $a \in S$ such that $\mathrm{Cone}(S \setminus \{a\}) \notin \Sigma$. In the latter case, we must have $a \notin \mathscr{P}$ to avoid violating the induction hypothesis. Without loss of generality we can take $a = v_s$. Repeatedly applying this line of reasoning, we find that $\{u_{i_0}, \ldots u_{i_r}\}$ does not generate a cone in Σ, which is only possible when $r = k$ as $\mathscr{P}$ is a primitive collection. Hence for any $A \subseteq \mathscr{P}$ with $|A| \leq k$, the set $A \cup \{v_0, \ldots, v_s\}$ generates a cone in Σ. That is, given two cones $\tau_1, \tau_2 \in \Sigma$ with generators all in $\mathscr{P}$ and all in $G(\Sigma) \setminus \mathscr{P}$ respectively, the cone $\tau_1 + \tau_2 = \{w_1 + w_2 \mid w_i \in \tau_i\}$ is also in Σ.

Conversely, consider a cone $\sigma \in \Sigma$. Then we can write

$$\sigma = \mathrm{Cone}(u_{i_0}, \dots u_{i_r}, v_0, \dots, v_s)$$

where $u_{i_0}, \dots u_{i_r} \in \mathscr{P}$ and $v_0, \dots v_s \in G(\Sigma) \setminus \mathscr{P}$. Then $\tau_1 = \mathrm{Cone}(u_{i_0}, \dots, u_{i_r}) \in \Sigma$ and $\tau_1 = \mathrm{Cone}(v_0, \dots, v_s) \in \Sigma$ as they are both faces of σ, and we have $\sigma = \tau_1 + \tau_2$.

Step 3. Motivated by the above, we define

$$\tilde{\Sigma} = \{\mathrm{Cone}(S) \in \Sigma \mid S \subseteq G(\Sigma) \setminus \mathscr{P}\}.$$

This collection of cones is a (not necessarily complete) fan as it inherits the fan properties from Σ. Define $\Sigma_B \subseteq (N_B)_\mathbb{R}$ to be the collection consisting of the images of the cones of $\tilde{\Sigma}$ by h. (Note that Σ_B does not change if we replace $\tilde{\Sigma}$ with Σ in this definition.) It is straightforward linear algebra to prove that Σ_B is a complete fan in $(N_B)_\mathbb{R}$. Note that the definitions of h, N_B, and Σ_B indeed make h a map of fans.

We can similarly define

$$\Sigma_F = \{\mathrm{Cone}(S) \subseteq \ker h = (N_F)_\mathbb{R} \mid S \subsetneq \mathscr{P}\}.$$

This collection of cones inherits the fan properties from Σ, and as $\mathscr{A} \subseteq \mathscr{P} = G(\Sigma_F)$, it is complete in $(N_F)_\mathbb{R}$.

Step 4. Fix $\sigma_B \in \Sigma_B$. By construction, this cone is the image of at least one cone in $\tilde{\Sigma}$. We claim that there is a unique $\tau \in \tilde{\Sigma}$ of same dimension as σ_B such that $h(\tau) = \sigma_B$.

To see this, suppose for contradiction that there exist $\tau_1, \tau_2 \in \tilde{\Sigma}$, both of same dimension as σ_B, such that $h(\tau_1) = h(\tau_2) = \sigma_B$. Then there is a generator v_1 of τ_1 and a generator v_2 of τ_2 such that $v_2 = v_1 + u$ for some $u \in N_F$. As Σ_F is complete in $(N_F)_\mathbb{R}$, there is a set $A \subseteq \mathscr{P}$ such that $u \in \mathrm{Cone}(A)$. However, by step 2, we have $\mathrm{Cone}(A \cup \{v_1\}) \in \Sigma$; this cone contains v_2 but $A \cup \{v_1\}$ does not, so we must have $v_2 \notin G(\Sigma)$. This is a contradiction. We can thus lift σ_B uniquely to τ, so (O2) holds with $\tilde{\Sigma}_B = \tilde{\Sigma}$.

This along with steps 2 and 3 proves (O3), so X_Σ is a toric fibre bundle by Proposition 4.3. Finally, condition (i) implies that Σ_F is the fan of $\mathbb{P}^k$ (see Section 2.3), and so X_Σ is a toric $\mathbb{P}^k$-bundle over the toric variety B. □

4.1 Decomposable Bundles

We can refine the statement of Theorem 4.4 if we require the second condition to hold for every pair of primitive collections. We call such a fan a *splitting fan*.

Theorem 4.5 ([Bat1, Theorem 4.3])**.** *Let Σ be a splitting fan. Then X_Σ is a projectivisation of a decomposable vector bundle over a toric variety B that is associated with a splitting fan of smaller dimension.*

The statement of this theorem uses several new concepts. We have already seen fibre bundles in general, but here we care about *vector bundles*. A vector bundle of rank n is a fibre bundle with fibre $F = \mathbb{C}^n$ as a complex vector space, and such that the transition functions are invertible linear maps $F \to F$. A vector bundle of rank 1 is called a *line bundle*.

Let V be a vector bundle. A (local) *section*[5] of V over an open set U of the base space is a morphism $s : U \to V$ satisfying $p \circ s(x) = x$ for all $x \in U$. A *global section* is a section whose domain is the whole base space.

Similar terminology exists for sheaves – given a sheaf $\mathcal{F}$ on a variety X, we call the elements of $\mathcal{F}(U)$ *sections* for any open set $U \subseteq X$. The elements of $\mathcal{F}(X)$ are called *global sections*.

This suggestive terminology implies a link between sheaves and bundles, and indeed there is one – the sections of any vector bundle over a variety X form a sheaf of $\mathcal{O}_X$-modules. It is common to identify a vector bundle with its associated sheaf.

Furthermore, we have the following link between line bundles and divisors, which we state without proof.

Proposition 4.6. *For any Cartier divisor D on a variety X, the sheaf $\mathcal{O}_X(D)$ is the sheaf of sections of some line bundle over X.*

(In fact, the converse is also true!) The idea is that if a Cartier divisor D coincides with $\operatorname{div}(f_i)$ on $U_i \subseteq X$ and $\operatorname{div}(f_j)$ on $U_j \subseteq X$, then f_j/f_i is an invertible regular function $U_i \cap U_j \to \mathbb{C}$ that can be used in conjunction with the gluing isomorphisms defining X to glue $U_i \times \mathbb{C}$ with $U_j \times \mathbb{C}$; this can be done for all sets of an open cover of X to construct a line bundle over X. The map f_j/f_i is then a transition function for this line bundle.

This gives us another way of understanding the sheaf $\mathcal{O}_{\mathbb{P}^k}(a)$ from Section 2.6.

Example 4.7 ([Ser, Ch. III §2])**.** Equip $\mathbb{P}^k$ with homogeneous coordinates $[t_0 : \cdots : t_k]$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{\mathbb{P}^k}$-modules. Let U_i, U_j be the open sets given by $t_i \neq 0$ and $t_j \neq 0$ respectively, and denote by $\mathcal{F}_i$ (resp. $\mathcal{F}_j$) the restriction of $\mathcal{F}$ to U_i (resp. to U_j). Then, as with fibre bundles, there are two ways of looking at $\mathcal{F}(U_i \cap U_j)$. However, unlike before, there are numerous ways to switch between these two points of view.

[5] The name comes from the idea that such a function carves out a "cross-section" of the total space V.

Fix $a \in \mathbb{Z}$. Then the map

$$\theta_{ij}(a) : \mathcal{F}_i(U_i \cap U_j) \to \mathcal{F}_j(U_i \cap U_j)$$
$$f \mapsto \frac{t_j^a}{t_i^a} f$$

is a well-defined isomorphism, as t_j/t_i is an invertible regular function on $U_i \cap U_j$. These maps act like gluing maps, i.e. they satisfy $\theta_{ij}(a) \circ \theta_{jk}(a) = \theta_{ik}(a)$ on $U_i \cap U_j \cap U_k$, and so we can use them to glue together the restrictions into a new sheaf, which we call $\mathcal{F}(a)$ (see [Ser, Ch. I §1]). We think of $\theta_{ij}(a)$ as "twisting" the restricted sheaves a times before gluing them together; indeed not twisting at all just reconstructs the original sheaf, as $\mathcal{F}(0) = \mathcal{F}$.

Applying this with $\mathcal{F} = \mathcal{O}_{\mathbb{P}^k}$ gives us the sheaf $\mathcal{O}_{\mathbb{P}^k}(a)$, and this coincides with the definition from Section 2.6. $\lozenge$

There are two more concepts to introduce before we can prove Theorem 4.5. The first is that of a *decomposable vector bundle* – this is a (sheaf corresponding to a) vector bundle which can be written as a direct sum of (sheaves corresponding to) line bundles.

The last concept is the *projectivisation* of a bundle. Given a vector bundle V, we define a projective bundle – its projectivisation $\mathbb{P}(V)$ – by taking the projective spaces of the fibres, i.e. replacing each copy of $\mathbb{C}^n$ with a copy of $\mathbb{P}^n$ (with the transition functions and trivialisation isomorphisms induced by those of V). If $\mathcal{V}$ is the sheaf associated with the vector bundle V, then the projectivisation $\mathbb{P}(\mathcal{V})$ of the sheaf is defined to be the projectivisation of the dual bundle of V, which is obtained by taking the duals of the fibres – the reason for this is explained in [CLS, pp. 316-317].

We are now in a position to prove the theorem. The following proof is adapted from [Bat1].

Proof of Theorem 4.5. The requirement that Σ is a splitting fan immediately fulfils condition (ii) of Theorem 4.4. Hence, to show that X_Σ is a projective bundle, it is sufficient to show that Σ has a primitive collection with zero focus. By Proposition 3.6, any torus-invariant divisor of X_Σ is also associated with a splitting fan. This allows us to use induction.

Suppose for contradiction that no primitive collection in $G(\Sigma)$ has zero focus. Fix $u_0 \in G(\Sigma)$, and let $\{\bar{u}_1, \ldots, \bar{u}_k\}$ be a primitive collection in $G(\Sigma_{u_0})$ with zero focus – this is our induction hypothesis. Then we apply Proposition 3.6.

Suppose $\mathscr{P} = \{u_0, \ldots, u_k\}$ is a primitive collection in $G(\Sigma)$. Then by construction there is an integer a such that

$$u_0 + \cdots + u_k = au_0.$$

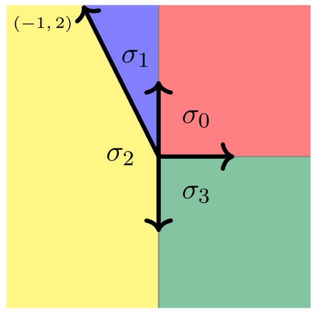

Figure 3: The fan of the Hirzebruch surface $\mathbb{F}_2$. To get the fan of any Hirzebruch surface $\mathbb{F}_a$ for some $a \in \mathbb{Z}$, replace the labelled generator with $(-1, a)$.

Either $a = 0$ – in which case we are done – or, by Proposition 3.4, we have $a \leq -1$. In the latter case, grouping the u_0 terms on the right and dividing by $\frac{a}{a-1} > 0$ shows that the sum $u_0 + \cdots + u_k$ lies within $\mathrm{Cone}(u_1, \ldots, u_k)$. The focus $\sigma(\mathscr{P})$ must then be a face of this cone, but this implies $\mathscr{P} \cap \sigma(\mathscr{P}) \neq \varnothing$, contradicting Proposition 3.4. Therefore $\{u_1, \ldots, u_k\}$ must be a primitive collection.

Now, for every generator $u \in G(\Sigma)$, we may assume there is a primitive collection, not containing u, with 1-dimensional focus. However, as every primitive collection is disjoint and has at least two elements, there are at most $\frac{1}{2}|G(\Sigma)|$ primitive collections in $G(\Sigma)$, so some of these must coincide. Thus there exists a primitive collection $\mathscr{P} = \{u_1, \ldots u_k\}$ in $G(\Sigma)$ and $v_1, v_2 \in G(\Sigma) \setminus \mathscr{P}$ such that $u_1 + \cdots + u_k$ is both an integer multiple of v_1 and an integer multiple of v_2. This is only possible if $v_2 = -v_1$, in which case $\{v_1, v_2\}$ is a primitive collection with zero focus.

By Theorem 4.4 we have that X_Σ is a projective bundle – write $X_\Sigma = \mathbb{P}(\mathcal{V})$ for some vector bundle $\mathcal{V}$. Then the fact that $\mathbb{P}(\mathcal{V})$ is a toric variety implies that $\mathcal{V}$ is decomposable, as remarked in [Oda, p31]. $\square$

Example 4.8. The family of Hirzebruch surfaces $\{\mathbb{F}_a\}_{a \in \mathbb{Z}}$ provides a good class of examples on which to demonstrate these theorems. The fan of $\mathbb{F}_2$ is shown in Figure 3. The observation that $\mathbb{F}_0$ is exactly $\mathbb{P}^1 \times \mathbb{P}^1$ – the trivial $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ – indicates that these may be projective bundles.

Indeed, we have exactly two primitive collections: $\mathscr{P}_1 = \{e_2, -e_2\}$ and $\mathscr{P}_2 = \{e_1, -e_1 + ae_2\}$, and these are disjoint. The fan of $\mathbb{F}_a$ is thus a splitting fan. Projection onto $\mathbb{R}^2/\operatorname{span}\{e_2\}$ gives us a 1-dimensional complete fan with generators $\pm e_1$, which is exactly the fan of $\mathbb{P}^1$. As the primitive collection $\mathscr{P}_1$ contains two elements, we have that $\mathbb{F}_a$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ by Theorem 4.4, and in particular we have $\mathbb{F}_a = \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$ for some line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ over $\mathbb{P}^1$ by Theorem 4.5.

In fact, we have $\mathbb{F}_a = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$. To see this, we consider how the affine pieces U_{σ_i} are glued together. Let $[t_0 : t_1]$ be homogeneous coordinates on $\mathbb{P}^1$, let

$x = t_1/t_0$, and let $U_0 \cong \mathbb{C}$ and $U_1 \cong \mathbb{C}$ be the open sets given by $t_0 \neq 0$ and $t_1 \neq 0$ respectively. Gluing U_{σ_0} to U_{σ_3} gives $U_0 \times \mathbb{P}^1$ in coordinates $(x, [u_0 : u_1])$ where $y = u_1/u_0$. Similarly, gluing U_{σ_1} to U_{σ_2} gives another copy of $U_1 \times \mathbb{P}^1$, in coordinates $(x^{-1}, [v_0 : v_1])$ where $x^a y = v_1/v_0$. The gluing isomorphism between these two copies of $\mathbb{C} \times \mathbb{P}^1$ is given by

$$\begin{aligned}(U_0 \cap U_1) \times \mathbb{P}^1 &\to (U_0 \cap U_1) \times \mathbb{P}^1 \\ (x, y) &\mapsto (x^{-1}, x^a y).\end{aligned}$$

The x components glue together into the base space $\mathbb{P}^1$, but the fibres in y have been "twisted" through multiplication by $x^a = t_1^a/t_0^a$ before being glued together.

We have used the transition functions for $\mathcal{O}_{\mathbb{P}^1}(0) = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(a)$ given in Example 4.7 as gluing isomorphisms in the construction of $\mathbb{F}_a$, as outlined in the sketch proof of Proposition 4.6. $\diamondsuit$

5 Classifying Toric Fano Varieties

Fano varieties have been studied by geometers for over a century. Of particular interest to modern algebraic geometry is their appearance in the minimal model program, which aims to facilitate the birational classification of algebraic varieties (a weaker notion than the one we discuss here). One outcome of the minimal model program is a Mori fibre space, whose fibres are Fano varieties. Thus to learn about algebraic varieties more broadly it is useful to start with a good understanding of Fano varieties.

In this section we discuss the classification of toric Fano varieties up to (biregular) isomorphism. We begin with the following definition.

Definition 5.1. A *Fano variety* is a smooth algebraic variety with ample anticanonical bundle.

Our goal in the following subsection is to clarify what it means for a toric variety to satisfy this condition.

5.1 Toric Fano Varieties

We saw above that line bundles are linked to Cartier divisors. In the case of a toric variety $X = X_\Sigma$, the *anticanonical bundle* is the line bundle (equivalently the sheaf) associated with the Cartier divisor

$$-K_X = \sum_{\rho \in \Sigma(1)} D_\rho.$$

We call $-K_X$ the *anticanonical divisor*.

The anticanonical bundle is *ample* precisely when the anticanonical divisor is ample. We must therefore discuss what it means for a divisor on a toric variety to be ample. To do this we use a special class of function.

Definition 5.2. A *support function* on a toric variety X_Σ is a function $\varphi : N_\mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(i) $\varphi(N) \subseteq \mathbb{Z}$.

(ii) φ is linear on each $\sigma \in \Sigma$.

We say φ is *strictly convex* if, for all $u, v \in N_\mathbb{R}$ not both lying in the same cone of Σ, we have

$$\varphi(u+v) > \varphi(u) + \varphi(v).$$

Equivalently (see [Ful, p. 68]) φ is strongly convex if the restrictions $\varphi|_{\sigma_1}$ and $\varphi|_{\sigma_2}$ are different linear functions for any distinct maximal cones $\sigma_1, \sigma_2 \in \Sigma$.

Every Cartier divisor on X_Σ gives us a support function in the following way. Note first that, by linearity, the restriction of any support function to a cone is uniquely defined by the values it takes on the generators of that cone, so to specify a support function on $N_\mathbb{R}$ we simply have to specify its values at the fan generators. Every generator $u \in G(\Sigma)$ generates a unique ray $\rho_u \in \Sigma(1)$, and as every Cartier divisor (up to linear equivalence) can be written in the form $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, we can define the support function φ_D by setting

$$\varphi_D(u) = -a_{\rho_u} \quad \forall u \in G(\Sigma).$$

We then say that D is *ample* when φ_D is strictly convex. Thus the anticanonical divisor is ample if and only if the support function taking the value -1 at each fan generator is strictly convex.

In the language of primitive collections, we now have the following necessary and sufficient condition for when a toric variety is a Fano variety.

Proposition 5.3 ([Bat2, Proposition 2.3.6])**.** *A toric variety X_Σ is a Fano variety if and only if every primitive collection $\mathscr{P} \subseteq G(\Sigma)$ satisfies $D(\mathscr{P}) > 0$.*

Proof. Let φ be the support function associated with the anticanonical divisor of X_Σ, and let $\mathscr{P}$ be a primitive collection with associated primitive relation

$$u_1 + \cdots + u_k - a_1 v_1 - \cdots - a_m v_m = 0.$$

Applying φ to both sides gives

$$\varphi(u_1 + \cdots + u_k) - a_1\varphi(v_1) - \cdots - a_m\varphi(v_m) = 0$$

as $\varphi(0) = 0$ and as the v_j all lie in the same cone. From this, we get

$$\varphi(u_1) + \cdots + \varphi(u_k) - a_1\varphi(v_1) - \cdots - a_m\varphi(v_m) < \varphi(0) = 0$$

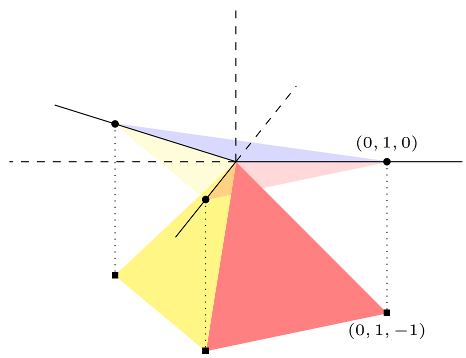

Figure 4: The graph of $\varphi_{(-K_X)}$ for the variety $X = \mathbb{P}^2$.

if and only if φ is strongly convex, i.e. if and only if X_Σ is a Fano variety. Finally, as the u_i and v_j are all fan generators, we have

$$-k - \sum_{j=1}^{m} a_j < 0.$$

Multiplying by -1 gives the desired result. □

Example 5.4. Recall the torus-invariant prime divisors of projective space $\mathbb{P}^k$ described in Example 2.15. The anticanonical divisor of $X = \mathbb{P}^k$ is $-K_X = \sum_{i=0}^{k} D_i$, so the anticanonical bundle is $\mathcal{O}_{\mathbb{P}^n}(k+1)$, with associated support function defined by $\varphi(\ell_i) = -1$ for all $i \in \{0, \ldots, k\}$.

The graph of this support function for $\mathbb{P}^2$ is shown in Figure 4. It is clear that φ defines a different linear function on each maximal cone, and hence is strictly convex. Thus $\mathbb{P}^2$ is a Fano variety.

Alternatively, we can use the fact that the only primitive collection for the fan of $\mathbb{P}^2$ is the set $\mathscr{P} = \{\ell_0, \ell_1, \ell_2\}$ consisting of all three fan generators. By definition of ℓ_0, the associated primitive relation is

$$\ell_0 + \ell_1 + \ell_2 = 0$$

and so the degree of this collection is $D(\mathscr{P}) = 3 > 0$. Thus $\mathbb{P}^2$ is a Fano variety. ◊

5.2 Fano Polytopes

In Section 2.4, we saw how to define a polytope using its facet presentation. Each lattice polytope is uniquely defined by the coefficients associated with the generators of its normal fan Σ, as are the Cartier divisors on X_Σ. This naturally gives rise to a correspondence between lattice polytopes and Cartier divisors.

We define the polytope[6] associated with a Cartier divisor $D = \sum_{i=1}^n a_i D_{\rho_i}$ using the facet presentation

$$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq -a_i \text{ for all } i \in \{1, \ldots, n\}\}$$

where $G(\Sigma) = \{u_1, \ldots, u_n\}$ and $\rho_i = \mathrm{Cone}(u_i)$. Note that the reverse of this process can be used to define a Cartier divisor using a polytope. When $X = X_\Sigma$ is a Fano variety, the polytope associated with its anticanonical divisor has facet presentation

$$P_{(-K_X)} = \{m \in M_{\mathbb{R}} \mid \langle m, u_i \rangle \geq -1 \text{ for all } i \in \{1, \ldots, n\}\}. \tag{5}$$

This polytope is clearly reflexive. Its dual $P \subseteq N_{\mathbb{R}}$ is thus also a reflexive lattice polytope, and satisfies the following conditions.

(i) The origin $0 \in N$ is in the interior of P.

(ii) Each face of P is a simplex.

(iii) For each facet F of P, the generators of F form a $\mathbb{Z}$-basis of N.

A lattice polytope satisfying (i)-(iii) is called a *Fano polytope*. They are so named for a very good reason.

Proposition 5.5. *If X is a toric Fano variety, then the dual polytope of $P_{(-K_X)}$ is a Fano polytope. Conversely, if P is a Fano polytope and Σ the fan of cones over its faces, then the toric variety X_Σ is a Fano variety.*

Proof. Let X be a toric Fano variety, and P the dual polytope to $P_{(-K_X)}$. As mentioned above, P is reflexive, so the only lattice point in its interior is $0 \in N$. By definition, X is smooth, and so the generators of each cone in its fan Σ form a subset of a $\mathbb{Z}$-basis of N. In particular, if d is the rank of N, the maximal cones of Σ have exactly d generators and hence these generators form a $\mathbb{Z}$-basis of N; these maximal cones are exactly the cones over the facets of P. Each k-dimensional face of P is a cross-section of some $(k+1)$-dimensional cone in Σ, so has $k+1$ vertices and is thus a simplex.

For the converse, let P be a Fano polytope, Σ the fan of cones over its faces, and $X = X_\Sigma$ the corresponding toric variety. Note that X is smooth due to conditions (ii) and (iii) from the definition of Fano polytopes. As P is reflexive, its dual $P' \subseteq M_{\mathbb{R}}$ is also a reflexive lattice polytope, so has facet presentation as in (5). The divisor on X associated with P' is clearly the anticanonical divisor.

We claim that the divisor associated with a full-dimensional lattice polytope is always ample. To see this, let $Q \subseteq M_{\mathbb{R}}$ be such a polytope and $D_Q = \sum_{i=1}^n a_i D_{\rho_i}$ the associated Cartier divisor on X. There is a support function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$

[6] Note that in general, this is simply a finite intersection of closed half-spaces (sometimes called a *polyhedron*) and may not be bounded, hence not a polytope. We are safe in this case because our fans are all assumed to be complete.

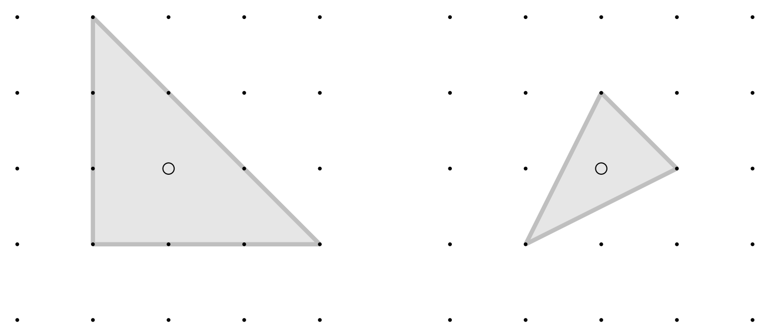

Figure 5: The polytope associated with the anticanonical divisor of $\mathbb{P}^2$ (left) and its dual polytope (right). The circle marks the origin.

defined by $\varphi(u_i) = -a_i$ for all $i \in \{1, \ldots, n\}$. Recall that every maximal cone $\sigma \in \Sigma$ corresponds to a vertex v_σ of Q. We have $\langle v_\sigma, u_i \rangle = -a_i$ for each i such that u_i is normal to a facet containing v_σ, i.e. for each i such that u_i is a generator of σ.

As $\langle v_\sigma, \cdot \rangle : N_\mathbb{R} \to R$ is linear, it coincides with φ on σ. Thus for any two maximal cones $\sigma_1, \sigma_2 \in \Sigma$ we have

$$\varphi|_{\sigma_1} = \varphi|_{\sigma_2} \Leftrightarrow v_{\sigma_1} = v_{\sigma_2} \Leftrightarrow \sigma_1 = \sigma_2$$

so φ is strictly convex, and D_Q is thus ample. Taking $Q = P'$ gives that X is a Fano variety. □

The consequence of this result is that, to classify toric Fano varieties up to isomorphism, it is sufficient (indeed equivalent) to classify Fano polytopes up to a suitable notion of "isomorphism" for lattice polytopes.

Example 5.6. Let's return to the example of $\mathbb{P}^2$. Call its fan generators $\ell_1 = (1,0), \ell_2 = (0,1)$ and $\ell_0 = (-1,-1)$. Then the equations $\langle m, \ell_i \rangle \geq -1$ define halfspaces $H_i \subseteq M_\mathbb{R}$, and the polytope associated with the anticanonical divisor of $\mathbb{P}^2$ is precisely the intersection of these halfspaces. We have:

$$\begin{aligned} H_0 &= \{(x,y) \in \mathbb{R}^2 \mid y \leq 1 - x\}, \\ H_1 &= \{(x,y) \in \mathbb{R}^2 \mid x \geq -1\}, \\ H_2 &= \{(x,y) \in \mathbb{R}^2 \mid y \geq -1\}, \end{aligned}$$

and obtain the triangle shown on the left in Figure 5. Looking now at the dual polytope on the right, it is clear that it satisfies conditions (i)-(iii) from the definition of a Fano polytope. Hence $\mathbb{P}^2$ is a Fano variety. ◊

5.3 When Are Two Fano Varieties Isomorphic?

There are two distinct notions of "sameness" we may wish to consider when approaching the classification of Fano polytopes. We call these *combinatorial equiva-*

lence and *isomorphism*. The following definitions and theorems are due to Batyrev [Bat2].

Definition 5.7. Let P_1 and P_2 be Fano polytopes in $N_{\mathbb{R}}$. We say that P_1 is *combinatorially equivalent* to P_2 if there is a bijection

$$\varphi : V(P_1) \to V(P_2)$$

which respects the face relation. That is, a set of vertices $\{v_1, \ldots, v_k\}$ of P_1 defines a face of P_1 if and only if $\{\varphi(v_1), \ldots \varphi(v_k)\}$ defines a face of P_2. In this case we say P_1 and P_2 have the same *combinatorial type*. More strongly, we say P_1 and P_2 are *isomorphic* if there is a linear map

$$\varphi : N_{\mathbb{R}} \to N_{\mathbb{R}}$$

such that $\varphi(N) = N$ and $\varphi(P_1) = P_2$.

Note that if P_1 and P_2 are isomorphic, then any linear map φ satisfying the above conditions restricts to a map of vertices respecting the face relation, so P_1 and P_2 are combinatorially equivalent. Moreover, such a φ is also a bijective map of fans (between the fans over the faces of P_1 and P_2 respectively), so it induces an isomorphism of Fano varieties.

As Batyrev points out, these notions are not captured by simply listing the vertices of a polytope – namely, such a list would give no indication of which vertices form faces. What we need are the primitive collections and relations defined in Section 3.

Proposition 5.8. *Two Fano polytopes P_1 and P_2 in $N_{\mathbb{R}}$ are combinatorially equivalent if and only if there is a bijection $\varphi : V(P_1) \to V(P_2)$ which induces a bijective correspondence between primitive collections in $V(P_1)$ and primitive collections in $V(P_2)$.*

The proof is straightforward and results from the fact that any proper subset of a primitive collection defines a face. Importantly, any map of vertex sets satisfying one definition of combinatorial equivalence also satisfies the other.

Proposition 5.9. *Two Fano polytopes P_1 and P_2 are isomorphic if and only if there is a bijection $\varphi : V(P_1) \to V(P_2)$ that induces a bijective correspondence between not only primitive collections in $V(P_1)$ and $V(P_2)$, but also their primitive relations.*

In the proof we make use of the following lemma from [Bat2], which we state without proof.

Lemma 5.10. *Let P be a d-dimensional Fano polytope with n vertices, and let $L(P)$ be the sublattice of $\mathbb{Z}V(P)$ consisting of $\mathbb{Z}$-linear relations amongst vertices of P. Then $L(P)$ is generated by primitive relations.*

Proof of Proposition 5.9. Suppose P_1 and P_2 are isomorphic and let $\varphi : N_{\mathbb{R}} \to N_{\mathbb{R}}$ be linear and satisfy $\varphi(N) = N$ and $\varphi(P_1) = P_2$. As mentioned above, φ preserves the face relation and thus primitive collections by Proposition 5.8. By bijectivity and linearity of φ, it also preserves primitive relations.

Conversely, suppose $\varphi : V(P_1) \to V(P_2)$ is a bijection which preserves primitive collections and relations. By Lemma 5.10, preserving primitive relations also preserves all linear relations amongst the vertices of $V(P_1)$. As $V(P_1)$ includes a basis of N, the map φ induces a linear map $\tilde{\varphi} : N_{\mathbb{R}} \to N_{\mathbb{R}}$ satisfying $\tilde{\varphi}(N) = N$. Moreover, since all points of a polytope are linear combinations of its vertices, we have $\tilde{\varphi}(P_1) = P_2$. □

Of course, if our goal is to classify toric Fano varieties up to isomorphism in a given dimension, it is wise to check that this is even reasonable – i.e. that there exist a finite number of isomorphism classes (or potentially families of isomorphism classes), so that we may list them all. This is indeed the case, as we will now show.

Lemma 5.11 ([Bat2, Proposition 2.1.11])**.** *If P is a d-dimensional Fano polytope, then $|V(P)| \leq 2(2^d - 1)$.*

Proof. Consider the quotient map

$$\alpha : N \cong \mathbb{Z}^d \to \mathbb{Z}^d/(2\mathbb{Z})^d.$$

No vertex v of P can have coordinates all divisible by 2, otherwise v could not be an element of a $\mathbb{Z}$-basis of N, so no vertex is in the kernel of α.

Fix two distinct vertices $u, v \in V(P)$ and assume that $\alpha(u) = \alpha(v) \neq 0$. Then $(u + v)/2$ is a lattice point, and by convexity it is contained in P. As P is reflexive, this point must either be a vertex or 0. The former cannot be true – if $(u + v)/2$ is on the boundary of P then it must lie in a face containing u and v, but this would contradict property (iii) of Fano polytopes – so we must have $u = -v$.

Thus for any nonzero $x \in \mathbb{Z}^d/(2\mathbb{Z})^d$, the fibre $\alpha^{-1}(x)$ contains at most two elements. There are $2^d - 1$ nonzero elements in $\mathbb{Z}^d/(2\mathbb{Z})^d$, so we get $|V(P)| \leq 2(2^d - 1)$. □

Lemma 5.12 ([Bat2, Proposition 2.1.10])**.** *Let P be a d-dimensional Fano polytope and let $\mathscr{P} = \{v_1, \ldots, v_k\} \subseteq V(P)$ be a primitive collection. Then there are only finitely many possibilities for the primitive relation associated with $\mathscr{P}$.*

Proof. By Proposition 5.3, we have $D(\mathscr{P}) = k - \sum_{i=1}^{m} a_i > 0$, i.e.

$$a_1 + \cdots + a_m < k.$$

We clearly have $k \leq d + 1$ as the faces of P have at most d vertices. All the a_i must be positive, so there are only finitely many options for $(a_1, \ldots, a_m)$. □

Combining these results we get the following theorem.

Theorem 5.13 ([Bat2, Theorem 2.1.13])**.** *For any fixed positive integer d, there are only finitely many d-dimensional Fano polytopes up to isomorphism.*

Proof. By Lemma 5.11 and the characterisation of combinatorial equivalence using primitive collections, there are finitely many combinatorial types of Fano polyhedra in dimension d. Furthermore, by Lemma 5.12, for a fixed combinatorial type and a fixed primitive collection, there are finitely many options for the associated primitive relation. Thus there are only finitely many possible primitive relations between the vertices of a d-dimensional Fano polytope. As primitive relations determine isomorphism classes, there are finitely many d-dimensional Fano polytopes up to isomorphism. □

5.4 Fano 4-folds with Picard Number 2

We have seen that isomorphism classes of toric Fano varieties are uniquely identified by the primitive relations on the corresponding Fano polytopes, and that it is feasible to list the isomorphism classes for a given dimension. Batyrev does exactly this in dimension 4 in [Bat2], from which we reproduce an example here.

Theorem 5.14. *There are exactly nine different 4-dimensional Fano polyhedra with six vertices, up to isomorphism.*

The proof of this theorem relies on a useful construction from [Kle]. Let s be an integer satisfying $2 \leq s \leq 4$ and let $r = 4 - s + 1$. Let $a_1 \leq \cdots \leq a_r$ be nonnegative integers. Our goal is to construct the fan of a toric variety, denoted $X_4(a_1, \ldots, a_r)$.

We define the following vectors in $\mathbb{R}^4$:

$$
\begin{aligned}
u_i &= e_i \text{ for all } i \in \{1, \ldots, r\}, \\
u_{r+1} &= -\sum_{i=1}^{r} e_i, \\
v_j &= e_{r+j} \text{ for all } j \in \{1, \ldots, s-1\}, \\
v_s &= \sum_{i=1}^{r} a_i e_i - \sum_{j=1}^{s-1} v_j.
\end{aligned}
$$

We then have a fan $\Sigma[a_1, \ldots a_r]$ with $G(\Sigma[a_1, \ldots, a_r]) = \{u_1, \ldots u_{r+1}, v_1, \ldots, v_s\}$ whose maximal cones are exactly those of the form

$$
\operatorname{Cone}(G(\Sigma[a_1, \ldots, a_r]) \setminus \{u_i, v_j\}); \quad i \in \{1, \ldots, r\}, j \in \{1, \ldots, s\}. \tag{6}
$$

We define $X_4(a_1, \ldots, a_r)$ to be the toric variety given by this fan.

Notice that $X_4(0) = \mathbb{P}^1 \times \mathbb{P}^3$ is isomorphic to $X_4(0,0,0) = \mathbb{P}^3 \times \mathbb{P}^1$. Importantly, this is the only duplicate from this construction. To see this, it is sufficient to note that as maps of fans are linear, and as $G(\Sigma[a_1,\dots,a_r])$ includes a basis of $\mathbb{R}^4$ that does not depend on $(a_1,\dots,a_r)$, no map of fans takes $\Sigma[a_1,\dots,a_r]$ to $\Sigma[b_1,\dots,b_{r'}]$ when $(a_1,\dots,a_r) \neq (b_1,\dots,b_{r'})$ unless $v_{s'} = u_{r+1}$, which only happens if all the a_i and the $b_{i'}$ are 0.

We then have the following result which, for the sake of brevity, we state without proof.

Lemma 5.15. *Every 4-dimensional toric variety with Picard number 2 is isomorphic to exactly one of the varieties $X_4(a_1,\dots,a_r)$, modulo identification of $X_4(0)$ with $X_4(0,0,0)$.*

Thus to classify toric Fano varieties with Picard number 2, it suffices to classify the $X_4(a_1,\dots,a_r)$ which are Fano varieties. Kleinschmidt's paper also gives us a criterion for when this is the case.

Lemma 5.16. *Let s, r, and $(a_1,\dots,a_r)$ be as above. The variety $X_4(a_1,\dots,a_r)$ is a Fano variety if and only if $\sum_{i=1}^r a_i < s$.*

Proof. By construction, $\mathscr{P}_1 = \{v_1,\dots,v_s\}$ is a primitive collection, with corresponding primitive relation

$$v_1 + \cdots + v_s - a_1u_1 - \cdots - a_ru_r = 0.$$

The degree of this primitive collection is $D(\mathscr{P}_1) = s - \sum_{i=1}^r a_i$. We also have the primitive collection $\mathscr{P}_2 = \{u_1,\dots,u_{r+1}\}$ with primitive relation

$$u_1 + \cdots + u_{r+1} = 0.$$

By (6), these are the only two primitive collections for this fan.[7]

We always have $D(\mathscr{P}_2) = r > 0$. Hence by Proposition 5.3, $X_4(a_1,\dots,a_r)$ is a Fano variety if and only if $s - \sum_{i=1}^r a_i > 0$. □

Proof of Theorem 5.14. By the above lemmas, the following ten cases give those $X_4(a_1,\dots,a_r)$ which are Fano varieties:

$$\begin{aligned} &s = 4, \quad a_1 \in \{0,1,2,3\};\\ &s = 3, \quad (a_1,a_2) \in \{(0,0),(0,1),(1,1),(0,2)\};\\ &s = 2, \quad (a_1,a_2,a_3) \in \{(0,0,0),(0,0,1)\};\end{aligned} \tag{7}$$

and since $(s = 4, a_1 = 0)$ and $(s = 2, (a_1,a_2,a_3) = (0,0,0))$ give isomorphic varieties, there are nine toric Fano 4-folds with Picard number 2 up to isomorphism. □

[7] This has as a consequence that every 4-dimensional toric variety of Picard number 2 is the projectivisation of a decomposable bundle, by Theorem 4.5. In fact, we can define varieties $X_d(a_1,\dots,a_r)$ for any $d \geq 2$ similarly to as above, and this holds for those varieties as well. In particular, for any $a \in \mathbb{Z}$, the variety $X_2(a)$ is exactly the Hirzebruch surface $\mathbb{F}_a$ from Example 4.8.

The cones of $\Sigma[a_1, \dots a_r]$ are exactly the cones over the faces of the Fano polytope P of $X_4(a_1, \dots, a_r)$, so the primitive relations of P are just $\mathscr{P}_1$ and $\mathscr{P}_2$ from the proof of Lemma 5.16. Since the focus of $\mathscr{P}_2$ is always 0, the values of $a_1, \dots, a_r$ given in (7) are all we need to describe the nine isomorphism classes using primitive relations. After renaming vertices to combine the cases $s = 4$ and $s = 2$ (our six vertices are now labelled v_1 through v_6) we get Table 1.

$v_1 + v_2 + v_3 + v_4 =$	0	v_5	$2v_5$	$3v_5$	0
$v_5 + v_6 =$	0	0	0	0	v_1

$v_1 + v_2 + v_3 =$	0	v_4	$v_4 + v_5$	$2v_4$
$v_4 + v_5 + v_6 =$	0	0	0	0

Table 1: The nine isomorphism classes of 4-dimensional Fano polytopes with six vertices, described using primitive relations. There are two distinct combinatorial types.

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